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The longitudinal deformation of a rod in friction interaction with a compressing body☆

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Abstract

The deformation of a rod, confined in a fixed external housing, is considered. The friction forces in the contact surface are related to the deformation of the rod by a power relation. A wide range of variation of the friction parameter and the preliminary clearance parameter with which the rod is inserted into the housing is investigated and the characteristic features of the stress and strain distributions are revealed. The dissipation of energy due to friction and the formation of a hysteresis loop in the dependence of the stresses in the loaded end face on its displacement are considered. The problem is solved in a quasistatic formulation. Analytical relations are found for a number of important cases. Other results are obtained by numerical integration of the initial differential problem.

The model of a rod confined in an external housing enables us to describe many mechanical objects, the characteristic feature of which is the dependence of the friction in the lateral surface on their expansion and contraction accompanying longitudinal deformation. Such a structural model corresponds, for example, to the conditions under which the rubber inserts in specific types of shock absorbers, multi-strand wires in a braid and cables with compression casings operate. The wide introduction of cable systems in space technology makes this problem of particular current interest. Finally, the loading of casing tubes and the drawing of a filament through the thickness of a material can be investigated on the basis of this problem. There is considerable interest in the scattering of energy accompanying the frictional interaction between a rod and a compressing body, since it is precisely this which determines the damping of oscillations.

In spite of the wide range of designs being modelled, the mechanics of a rod, which interacts by friction with an external housing, has not been studied sufficiently. Papers dealing with this problem only touch on cases when the friction forces are constant or directly proportional to the stresses in the rod, which is infinite in one direction.[1–4] In this paper, the class of possible friction stresses in the contact surface is extended considerably and the length of the rod is assumed to be finite.

1. Formulation of the problem

The stress-strain state of a thin elastic rod of constant cross-section when it interacts with a fixed rigid housing is described by the following system of equations

$$\sigma' = p, \quad u' = \sigma \tag{1.1}$$

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Here σ and u are the stresses in the cross-sections and the displacements of the cross-sections themselves along the longitudinal axis, a prime denotes differentiation with respect to the axial coordinate x, and p is the reduced friction stress between the rod and the housing. The stresses are divided by the modulus of elasticity E, and the displacements and the x coordinate are divided by the length of the rod l. Then, $p = q(l/EF)$, where q is the friction stress per unit length on the lateral surface of the rod and F is the cross-section area.

In defining the conditions for the interaction between the housing and the rod, we will assume that the friction force obeys the Amonton - Coulomb law and that, in the general case, the normal pressure depends on the settling of the rod in a non-linear manner. Such a situation arises, for example, when the lateral face is uneven and the contact area changes during the deformation process. Suppose the normal pressure changes according to a power law as the rod is deformed. The friction interaction can then be described as follows:

$$\sigma' = |p|k, \quad |p| = \xi\left|g + \eta|u'|^{\theta+1}\operatorname{sign}u'\right|$$

Here g is the pressure created due to the precompression of the rod by the housing, ξ is the friction coefficient and η and θ are parameters which relate the normal pressure to the longitudinal strain of the rod. Generally speaking, the parameter k can take any value with a modulus no greater than unity. Unlike the case of an absolutely rigid body, any change in the axial force in a deformable rod with plane cross sections is accompanied by a slippage of the lateral face with respect to the fixed mounting. We shall assume that the force changes monotonically both during loading and unloading. In the first case $k = 1$ and, in the second case, $k = -1$.

Contact between the rod and the housing is maintained while $g + \eta|u'|^{\theta+1}\operatorname{sign}u' < 0$. On taking account of this and the relation between the elastic strains and stresses, we can write the equations for the equilibrium of the rod in the final form

$$\sigma' = \begin{cases} -k\Phi(\sigma), & \Phi(\sigma) < 0 \\ 0, & \Phi(\sigma) \geq 0 \end{cases} \tag{1.2}$$

$$\Phi(\sigma) = f + \lambda|\sigma|^{\theta+1}\operatorname{sign}\sigma, \quad f = \xi g, \quad \lambda = \xi\eta$$

The second equation of system (1.1) is used to determine the displacements. We will assume that the external axial force creates a stress $\sigma(0)$ at the end $x = 0$ and that the other end of the rod is securely clamped: $u(1) = 0$. There are no stresses and displacements in the rod prior to the application of the external load.

2. Theoretical relations

We shall first consider the case when the rod is inserted into the housing freely without any preliminary tightness. Contact and friction interactions arise when the rod is compressed and there are no such interactions when it is stretched. The condition for the maintenance of contact takes the form sign $\sigma < 0$, and the equilibrium equations can be written as follows:

$$\sigma' - k\lambda(-\sigma)^{\theta+1} = 0, \quad \operatorname{sign}\sigma < 0, \quad \sigma' = 0, \quad \operatorname{sign}\sigma \geq 0 \tag{2.1}$$

To begin with, the first equation will be considered, corresponding to the existence of contact. When $\theta = 0$ and k has a constant value, it is a linear equation. On taking account of the assumed boundary conditions for the first compression of the rod and $\sigma(0) = -a$, we obtain

$$\sigma = -ae^{-\lambda x}, \quad u = \frac{a}{\lambda}(e^{-\lambda x} - e^{-\lambda}) \tag{2.2}$$

During unloading, when the stresses on the end face $x = 0$ decrease from their maximum value a_m to zero, an unloading front is observed with the coordinate x_-. For brevity, we shall refer to the boundaries separating segments of the rod with different states as fronts. Of course, this term is conditional in a quasistatic problem and, at the same time, the above-mentioned boundaries manifest certain properties which make them like real fronts. For example, they move when the load changes.

If the discharge front remains within the limits of the rod, then

$$\sigma = -ae^{\lambda x}, \quad u = \frac{1}{\lambda}(2\sqrt{aa_m} - a_m e^{-\lambda} - ae^{\lambda x}), \quad x \le x_-$$
$$\sigma = -a_m e^{-\lambda x}, \quad u = \frac{a_m}{\lambda}(e^{-\lambda x} - e^{-\lambda}), \quad x_- < x \le 1 \tag{2.3}$$

$$x_- = \frac{1}{2\lambda}\ln\frac{a_m}{a} \le 1$$

If $x_- > 1$, then the expression for σ remains as before and corresponding to the first of the domains shown above, and the displacements in the whole of the rod are described by the second formula of (2.2) in which λ has been replaced by $-\lambda$.

In accordance with what has been said, when $\theta = 0$, there are no residual stresses and displacements after the external load has been removed. The energy dissipated after a single loading cycle is determined by the work done out by the external force on the end face $x = 0$. On referring it to the maximum energy of the elastic strain of the rod, we obtain the absorption coefficient, which is frequently used when estimating the damping of oscillations.[4] In the case being considered, this coefficient has the form

$$\psi = \frac{2}{3\lambda}\left[1 - \frac{1}{2}(3e^{-\lambda} - e^{-3\lambda})\right]$$

A similar interaction was considered previously in Ref. 1 in the case of a semi-infinite rod.

We will now consider the case when $\theta \neq 0$. The solution of the first equation of system (2.1) has the form

$$\sigma = -ar^{-1/\theta}, \quad r = 1 + \theta k\lambda a^{\theta} x \tag{2.4}$$

where account has been taken of the boundary condition on the end face $x = 0$.

On integrating equality (2.4) with respect to x, we obtain an expression for the displacements

$$u = \frac{a^{1-\theta}}{k\lambda(1-\theta)} r^{(\theta-1)/\theta} + C \tag{2.5}$$

The constant C is determined from the corresponding boundary conditions.

If the parameter α, which is defined by the relation

$$\alpha = a\lambda^{1/\theta} \tag{2.6}$$

is introduced into the solution, we obtain the expressions

$$\sigma\lambda^{1/\theta} = -\alpha r_\alpha^{-1/\theta}, \quad u\lambda^{1/\theta} = \frac{\alpha^{1-\theta}}{k(1-\theta)} r_\alpha^{(\theta-1)/\theta} + C$$

$$r_\alpha = 1 + \theta k\alpha^{\theta} x$$

Consequently, if the loading scale is changed by multiplying it by $\lambda^{1/\theta}$, the scale of the solution changes in exactly the same way.

It is obvious that

$$\frac{\sigma}{a_m} = \frac{\sigma}{\alpha_m}\lambda^{1/\theta}, \quad \frac{u}{a_m} = \frac{u}{\alpha_m}\lambda^{1/\theta}, \quad \alpha_m = a_m\lambda^{1/\theta}$$

The stresses and displacements, normalized with respect to a_m, can be expressed for a given value $\theta \neq 0$ in terms of the single parameter α, which reflects the combined effect of loading and friction, and the magnitude of ψ is determined by the value of α_m, which corresponds to the maximum load. The parameter α cannot be used when $\theta = 0$. However, in this case, ψ also depends solely on a single parameter λ which characterizes the level of friction.

We will first consider the interval $\theta < 0$. In the case of compression, when $k = 1$ and $\theta < 0$, the quantity r can change sign. If it becomes negative, u and σ only remain real for certain values of θ. On the other hand, the displacements increase as the external force becomes larger, that is, during the loading stage. The condition $\partial u/\partial a > 0$ must therefore be satisfied in the case of the compression of the rod and $k = 1$. On differentiating relation (2.5), we obtain

$$\partial u/\partial a = 1/(k\lambda a^{\theta} r^{1/\theta})$$

When $r < 0$, the condition $\partial u/\partial a > 0$ is also only satisfied for specific values of $\theta < 0$. Finally, if $r < 0$, relation (2.4) when $\theta = -1$, that is, in the case of constant dry friction, presupposes the possibility of the occurrence of stretching zones during the course of the first compression, although it is well known that there are no such zones.[1,4] At the same time, a real solution must exist for any $\theta < 0$. The same also applies to the condition $\partial u/\partial a > 0$. When $\theta = -1$, the known results for constant friction must be obtained. It is therefore logical to assume that the branch of the solution which corresponds to $r < 0$ does not have a physical meaning and must be discarded. When $r > 0$, the above-mentioned contradictions do not arise. It follows from the condition that the minimum value of r should be equal to zero that the domain of propagation of perturbations is finite during the compression of the rod and it is limited to the interval $0 \le x \le x_+$, where $x_+ = -a^{-\theta}/\theta$ is the coordinate of the first compression front which is formed here as in the case when $\theta = -1$.

Next, to be specific, we shall assume that $\theta = -1/2$. Then, by formula (2.6), $\alpha = \alpha/\lambda^2$ and $\alpha_m = a_m/\lambda^2$.

We now introduce the notation: $A_x = \sqrt{\alpha} + x/2$, $B_x = \sqrt{\alpha_m} + x/2$.

During the first compression, if $x_+ = 2\sqrt{\alpha} \le 1$, then

$$\sigma = -\lambda^2 A_{-x}^2, \quad u = \frac{2}{3}\lambda^2 A_{-x}^3, \quad x \le x_+; \quad \sigma = 0, \quad u = 0, \quad x_+ < x \le 1$$

If $x_+ > 1$, the expression for the stresses remains the same and the expression for the displacements takes the form

$$u = \frac{2}{3}\lambda^2 (A_{-x}^3 - A_{-1}^3), \quad 0 \le x \le 1$$

The domain of propagation of the perturbations depends on the maximum value of the external force and is bounded by the coordinate $x_{m+} = 2\sqrt{\alpha_m}$ when $x_{m+} < 1$. An unloading domain arises in the rod as the compression of the rod is reduced. It lies in the interval $0 < x \le x_-$. The coordinate of the unloading front x_- and the integration constant C are found from the conditions for the continuity of the stresses and displacements. In the unloading stage, $k = -1$ and, if $x_{m+} \le 1$, then

$$\sigma = -\lambda^2 A_x^2, \quad u = -\frac{2}{3}\lambda^2 \left[A_x^3 - \frac{1}{4}(\sqrt{\alpha} + \sqrt{\alpha_m})^3\right], \quad x \le x_-$$

$$\sigma = -\lambda^2 B_{-x}^2, \quad u = \frac{2}{3}\lambda^2 B_{-x}^3, \quad x_- < x \le x_{m+}$$

$$\sigma = 0, \quad u = 0, \quad x_{m+} < x \le 1; \quad x_- = \sqrt{\alpha_m} - \sqrt{\alpha}$$

If $x_{m+} > 1$, but $x_- \le 1$, then

$$\sigma = -\lambda^2 A_x^2, \quad u = -\frac{2}{3}\lambda^2 \left[A_x^3 - \frac{1}{4}(\sqrt{\alpha} + \sqrt{\alpha_m})^3 + B_{-1}^3\right], \quad x \le x_-$$

$$\sigma = -\lambda^2 B_{-x}^2, \quad u = \frac{2}{3}\lambda^2 (B_{-x}^3 - B_{-1}^3), \quad x_- < x \le 1$$

Finally, when $x_- > 1$,

$$\sigma = -\lambda^2 A_x^2, \quad u = -\frac{2}{3}\lambda^2 (A_x^3 - A_1^3), \quad 0 \le x \le 1$$

An analysis of the above relations shows that residual displacements and stresses are observed after the removal of the external load from the rod. The loops in the dependences of the stresses on the displacements in the end face $x = 0$ remain unclosed. On account of this, we shall consider a second compression of the rod. The second compression front is determined by the coordinate $x_{++} = \sqrt{\alpha}$. Its maximum distance from the loaded end is identical to the limiting position of the unloading front x_{m-}. Several versions of the stress and displacement distributions along the rod are possible depending on the past loading history.

If $x_{m+} \le 1$ and $x_{m-} \le 1$, then

$$\sigma = -\lambda^2 A_{-x}^2, \quad u = \frac{2}{3}\lambda^2\left[A_{-x}^3 - \frac{1}{4}(\alpha^{3/2} - \alpha_m^{3/2})\right], \quad x \le x_{++}$$

$$\sigma = -\frac{1}{4}\lambda^2 x^2, \quad u = -\frac{1}{6}\lambda^2\left(\frac{x^3}{2} - \alpha_m^{3/2}\right), \quad x_{++} < x \le x_{m-}$$

$$\sigma = -\lambda^2 B_{-x}^2, \quad u = \frac{2}{3}\lambda^2 B_{-x}^3, \quad x_{m-} < x \le x_{m+}$$

$$\sigma = 0, \quad u = 0, \quad x_{m+} < x \le 1$$

Upon reaching the maximum compression, the stresses and displacements take the same values as at the end of the first compression. A closed hysteresis loop is now formed in the $\sigma - u$ dependence for $x = 0$, which is repeated during successive loading cycles. The energy dissipation is characterized by the absorption coefficient

$$\psi = \frac{11}{30}\alpha_m \tag{2.7}$$

When $x_{m+} > 1$, the whole of the rod becomes deformed and the penultimate domain extends right up to $x = 1$. If $x_{m-} \le 1$, the stress distribution in the three remaining domains is described by the previous relations. However, the expressions for the displacements are changed, and we have

$$u = \frac{2}{3}\lambda^2\left[A_{-x}^3 - \frac{1}{4}(\alpha^{3/2} - \alpha_m^{3/2}) - B_{-1}^3\right], \quad x \le x_{++}$$

$$u = -\frac{2}{3}\lambda^2\left(\frac{x^3}{8} - \frac{\alpha_m^{3/2}}{4} + B_{-1}^3\right), \quad x_{++} < x \le x_{m-}$$

$$u = \frac{2}{3}\lambda^2(B_{-x}^3 - B_{-1}^3), \quad x_{m-} < x \le 1$$

In this case, the absorption coefficient is defined, as before, by relation (2.7).

When $x_{m-} > 1$, the repeated compression front, when the loading is increased, reaches the limits of the rod and the stress and displacement distributions depend solely on the position of this front.

So long as $x_{++} \le 1$,

$$\sigma = -\lambda^2 A_{-x}^2, \quad u = \frac{2}{3}\lambda^2\left[A_{-x}^3 - \frac{1}{4}\left(\alpha^{3/2} - \frac{1}{2}\right)\right], \quad x \le x_{++}$$

$$\sigma = -\frac{1}{4}\lambda^2 x^2, \quad u = \frac{1}{12}\lambda^2(1 - x^3), \quad x_{++} < x \le 1$$

After the repeated compression front disappears, by passing beyond the limits of the rod and $x_{++} > 1$,

$$\sigma = -\lambda^2 A_{-x}^2, \quad u = \frac{2}{3}\lambda^2(A_{-x}^3 - A_{-1}^3), \quad 0 \le x \le 1$$

If $x_{m-} > 1$, the absorption coefficient in the case of cyclic compression is given by the expression

$$\psi = \frac{1}{30\alpha_m^2}(40\alpha_m^{3/2} - 40\alpha_m + 15\alpha_m^{1/2} - 4)$$

We will now consider the case when $\theta > 0$ and put $\theta = 1$. Then, $\alpha = a\lambda$, $\alpha_m = a_m\lambda$. It has already been pointed out that expression (2.5) is unsuitable for describing the displacements for this value of θ.

In this case, the stress and displacement distributions correspond to the relations

$$\sigma = -\frac{\alpha}{\lambda(1+k\alpha x)}, \quad u = -\frac{1}{k\lambda}\ln(1+k\alpha x)+C$$

For the compression stage, we have

$$\sigma = -\frac{\alpha}{\lambda(1+\alpha x)}, \quad u = \frac{1}{\lambda}\ln\frac{1+\alpha}{1+\alpha x}, \quad 0 \le x \le 1$$

Here, there is no initial loading front as when $\theta = 0$. Note that this property is conserved for all values of $\theta > 0$ since $r > 0$ when $k = 1$ for any coordinates $x \ge 0$. We recall that this front is determined from the condition that $r = 0$.

If, after it has reached its maximum value, the load starts to be reduced and an unloading front arises in the rod with a coordinate $x_- = (\alpha_m - \alpha)/(2\alpha\alpha_m)$. In that part of the rod, where $x > x_-$, the stresses and displacements retain the magnitude reached during the compression stage and, when $x \le x_-$, an unloading process is observed. When $x_- < 1$,

$$\sigma = -\frac{\alpha}{\lambda(1-\alpha x)}, \quad u = \frac{1}{\lambda}\ln\frac{4\alpha\alpha_m(1+\alpha_m)(1-\alpha x)}{(\alpha+\alpha_m)^2}, \quad x \le x_-$$

$$\sigma = -\frac{\alpha_m}{\lambda(1+\alpha_m x)}, \quad u = \frac{1}{\lambda}\ln\frac{1+\alpha_m}{1+\alpha_m x}, \quad x_- < x \le 1$$

If $x_- \ge 1$ and the unloading front disappears, reaching beyond the limits of the rod, the formulae for the stresses and displacements in the unloading zone have the same form as in the compression stage but the quantities α and λ are replaced by $-\alpha$ and $-\lambda$.

After removal of the external load, there are no residual stresses and displacements in the rod. A closed hysteresis loop is now formed in the dependence of $\sigma(0)$ on $u(0)$ after the first loading - unloading cycle, and it remains unchanged during the course of subsequent loading. The energy dissipation is characterized by the following absorption coefficient

$$\psi = \frac{2}{\alpha_m^2}\left[2\alpha_m - \ln(1+2\alpha_m) + 2\alpha_m\ln\frac{1+\alpha_m}{1+2\alpha_m}\right]$$

We now return to the situation when, prior to the application of the load, the compressing body creates a normal pressure on the contact surface. For example, the rod is compressed by the housing which is tightly fitted onto it. Then, $f \ne 0$ in Eq. (1.2). We will assume that the external loading cycle starts with stretching.

As when $f = 0$, we begin with the case when $\theta = 0$. In this case, the solution of Eq. (1.2) has a fairly simple analytical expression. In the first stretching, when $0 \le a \le a_m$ and $x_+ \le 1$,

$$\sigma = (a+b)e^{\lambda x} - b, \quad u = b(x_+ - x) + \frac{a+b}{\lambda}\left(e^{\lambda x} - \frac{b}{a+b}\right), \quad x \le x_+$$

$$\sigma = 0, \quad u = 0, \quad x_+ < x \le 1$$

$$b = \frac{f}{\lambda}, \quad x_+ = \frac{1}{\lambda}\ln\frac{b}{a+b}$$

where x_+ is the coordinate of the initial loading front which is formed here unlike the case of a rod without precompression. If this front reaches beyond the limits of the rod, that is, $x_+ > 1$, the stresses are determined by the previous relation and the expression for the displacements takes the form

$$u = b(1-x) - \frac{a+b}{\lambda}(e^{\lambda x} - e^{\lambda}), \quad 0 \le x \le 1 \tag{2.8}$$

We shall henceforth assume that the stretching encompasses the whole of the rod. The extension of the results to the case of weaker stretching is quite obvious.

In the unloading and possible subsequent compression stage $a_m > a \geq a_n$, where it is possible that $a_n \leq 0$. The unloading front has the coordinate

$$x_- = \frac{1}{2\lambda}\ln\varsigma_-, \quad \varsigma_- = \frac{a+b}{a_m+b} \tag{2.9}$$

If a loss of contact between the rod and the compressing body is observed during the stretching stage, then $a_m + b \geq 0$, and the coordinate x_- cannot be determined since, according to formula (2.9), a division by zero occurs or the logarithm of a negative number has to be found. Physically, this means that an unloading front does not occur even after a temporary loss of contact.

When there is an unloading front and $x_- \leq 1$

$$\sigma = (a+b)e^{-\lambda x} - b$$

$$u = b(1-x) - \frac{a+b}{\lambda}\left(e^{-\lambda x} + \frac{e^{\lambda}}{\varsigma_-} - \frac{2}{\sqrt{\varsigma_-}}\right), \quad x \leq x_-$$

$$\sigma = (a_m+b)e^{\lambda x} - b$$

$$u = b(1-x) + \frac{a_m+b}{\lambda}(e^{\lambda x} - e^{\lambda}), \quad x_- < x \leq 1$$

If there is no unloading front in view of the fact that contact of the rod with the housing has disappeared or $x_- \geq 1$, the stresses in the whole of the rod are determined in the same way as in the first of the domains presented above. The displacements in this case are described by expression (2.8) in which λ is replaced by $-\lambda$.

An analysis of the relations which have been presented indicates that there are residual stresses and displacements up to the instant of the complete removal of the external force.

The limiting position of the compression front is determined by the coordinate

$$x_{n-} = \frac{1}{2\lambda}\ln\varsigma_{n-}, \quad \varsigma_{n-} = \frac{a_n+b}{a_m+b}$$

In the case of repeated stretching $a_n < a \leq a_m$ and a further front with the coordinate

$$x_{++} = \frac{1}{2\lambda}\ln\varsigma_{++}, \quad \varsigma_{++} = \frac{a_n+b}{a+b}$$

occurs. When both fronts remain within the limits of the rod, that is, $x_{++} \leq 1$ and $x_{n-} \leq 1$,

$$\sigma = (a+b)e^{\lambda x} - b$$

$$u = b(1-x) + \frac{a_n+b}{\lambda}\left(\frac{e^{\lambda x}}{\varsigma_{++}} - \frac{e^{\lambda}}{\varsigma_{n-}} - \frac{2}{\sqrt{\varsigma_{++}}} + \frac{2}{\sqrt{\varsigma_{n-}}}\right), \quad x \leq x_{++}$$

$$\sigma = (a_n+b)e^{-\lambda x} - b$$

$$u = b(1-x) - \frac{a_n+b}{\lambda}\left(e^{-\lambda x} + \frac{e^{\lambda}}{\varsigma_{n-}} - \frac{2}{\sqrt{\varsigma_{n-}}}\right), \quad x_{++} < x \leq x_{n-}$$

$$\sigma = (a_m+b)e^{\lambda x} - b$$

$$u = b(1-x) + \frac{a_m+b}{\lambda}(e^{\lambda x} - e^{\lambda}), \quad x_{n-} < x \leq 1$$

If $x_{n-} \geq 1$ or there is no unloading front at all, the second domain of the three domains mentioned above completely occupies the part of the rod with the coordinates $x > x_{++}$. The formulae for the stresses in the two remaining domains are as before, and the relations describing the displacements take the form

$$u = b(1-x) + \frac{a_n + b}{\lambda}\left(\frac{e^{\lambda x}}{\varsigma_{++}} + e^{-\lambda} - \frac{2}{\sqrt{\varsigma_{++}}}\right), \quad x \leq x_{++}$$

$$u = b(1-x) + \frac{a_n + b}{\lambda}(e^{-\lambda} - e^{-\lambda x}), \quad x_{++} < x \leq 1$$

Finally, when $x_{++} > 1$,

$$\sigma = (a+b)e^{\lambda x} - b, \quad u = b(1-x) + \frac{a+b}{\lambda}(e^{\lambda x} - e^{\lambda}), \quad 0 \leq x \leq 1$$

When $a = a_m$, the stresses and displacements reach the same values as during the first stretching. Later, during cyclic loading, the deformation pattern will repeat itself. The important special case, when $a_n \geq 0$ corresponds to the loading of filaments, cables, wires, etc.

When $\theta \neq 0$ and $f \neq 0$, it is either not possible to obtain analytical expressions for σ and u at all or rarely used higher transcendental functions have to be enlisted to do this. In these cases, the initial differential problem was integrated using numerical methods. In the case of Eq. (1.2), the Cauchy problem with the initial condition $\sigma(0)$ was solved by the fourth-order Runge-Kutta method and the required integrals for determining the displacements and the absorption coefficient were calculated using Simpson's formula.[5] A step size of 0.001 was used in all cases. Control calculations with a step size which was five times smaller as well as a comparison with the results obtained using the analytical relations showed that the results were identical to no less than four significant figures.

3. Analysis of the results and examples

The stresses and displacements in all of the illustrations were normalized with respect to the maximum value of the external load $a_m = 0.003$. The stress distribution in the rod placed in a housing without precompression is shown in Fig. 1 for the case of the action of a variable compressive force on the end $x = 0$. The friction interaction is characterized by the parameters $\theta = -1/2$ and $\lambda = 0.0775$ ($\alpha_m = 1/2$). The curves AC and DG describe the stress distribution in the first compression stage when $\sigma(0) = -a_m/2$ and $\sigma(0) = -a_m$ respectively. The loading front is located at the point C, which reaches beyond the limits of the rod when the compression is intensified. After unloading from $-a_m$ to $-a_m/2$, the stress level changes along the line AEG and the position of the unloading front is determined by the axial coordinate of

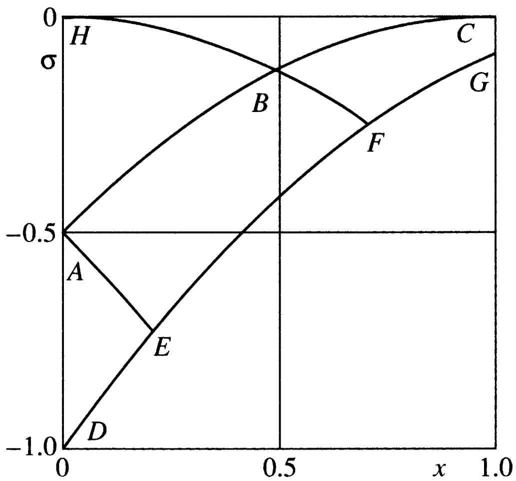

Fig. 1.

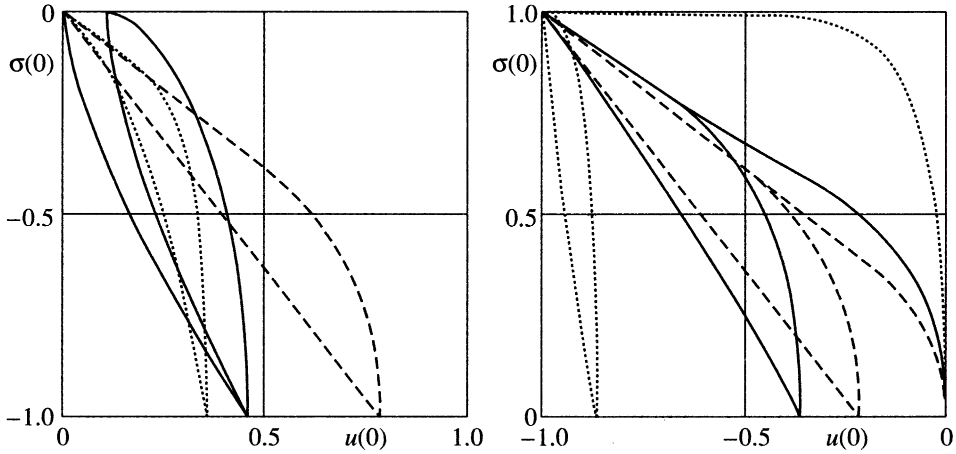

Fig. 2.

the point E. The curve HFG describes the stress distribution at the instant immediately preceding the complete removal of the external load when the unloading front reaches the coordinate of the point F.

The existence of an initial compression front as well as residual stresses and strains is, as was established in the preceding section, a characteristic feature of friction interaction of this type when $\theta < 0$.

On repeated compression up to half the maximum load, the stresses are distributed corresponding to the curve $ABFG$ and the secondary loading front moves from point B towards point F. On reaching the maximum load, it merges with the preceding unloading front, and the points B and F coincide. The stress distribution becomes the same as in the first compression, and, in the case of cyclic loading, the deformation pattern is repeated. We note that, even if the external load changes uniformly, the fronts are displaced at a variable rate. For example, the unloading front initially moves slowly and then its speed increases.

The formation of a hysteresis loop in the dependence of $\sigma(0)$ on $u(0)$ in the case of pulsating loading cycles is shown in Fig. 2. The results for a compressive load acting on the rod, which is placed in a housing without precompression, are shown on the left for $\theta = -1/2$ and $\lambda = 0.0775$ (the solid curve), $\theta = 0$ and $\lambda = 1$ (the dashed curve), and $\theta = 1$ and $\lambda = 1667$ ($a_m = 5$, the dotted curve) When $\theta = -1/2$, significant residual displacements are seen after the first unloading but the loop then becomes closed and stabilized. In the case when $\theta = 0$ and $\theta = 1$, there are no residual displacements and a closed hysteresis loop is already formed after the first cycle. When $\theta = 0$, the part of the loop corresponding to an increase in the friction force, exhibits a linear relation between $\sigma(0)$ and $u(0)$. In the two other cases, the part of the loop which is formed as the friction increases ceases to be rectilinear but its curvature is much less than the remaining part and has a different sign when $\theta = -1/2$ and $\theta = 1$.

The results of the stretching of a rod which is compressed in a housing are shown on the right-hand side of Fig. 2. The external load was varied over a range from 0 to a_m. Precompression was specified in accordance with the law $f = -\lambda|a_m|^{\theta+1}$, and complete opening of the joint between the rod and the housing was ensured at maximum extension.

When $\theta = 0$, if the first stretching, which is accompanied by the appearance of residual displacements, is discarded, the hysteresis loop has the same shape and size as in the case of pulsating compression. It is true that a linear dependence of the stress on the displacement corresponds here to the unloading stage rather than the loading stage. However, in both cases, it is observed when the friction force increases. When $\theta \neq 0$, there is no similar hysteresis loop as can be adjudged from the form of the other two loops. Their shape and dimensions are considerably different from those shown on the left-hand side of Fig. 2, although they were constructed for the same friction characteristics. However, here also, the dependence of the stress on the loaded end face on the displacement is closer to linear when the friction increases than when the friction decreases.

The stress distribution in a precompressed rod under the action of a load of alternating sign is shown in Fig. 3. The rod is first stretched to $\sigma(0) = a_m$ and then compressed to $\sigma(0) = -a_m$. The friction interaction when $\theta = 0$ and $\lambda = 1$ is considered as an example. The initial compression was specified in the same way as in the preceding case. The curve AB on the left-hand side corresponds to the first extension to $\sigma(0) = a_m/2$. On reaching $\sigma(0) = a_m$, contact between the rod and the housing disappears and identical stresses, which are equal to unity in the normalized form, are established

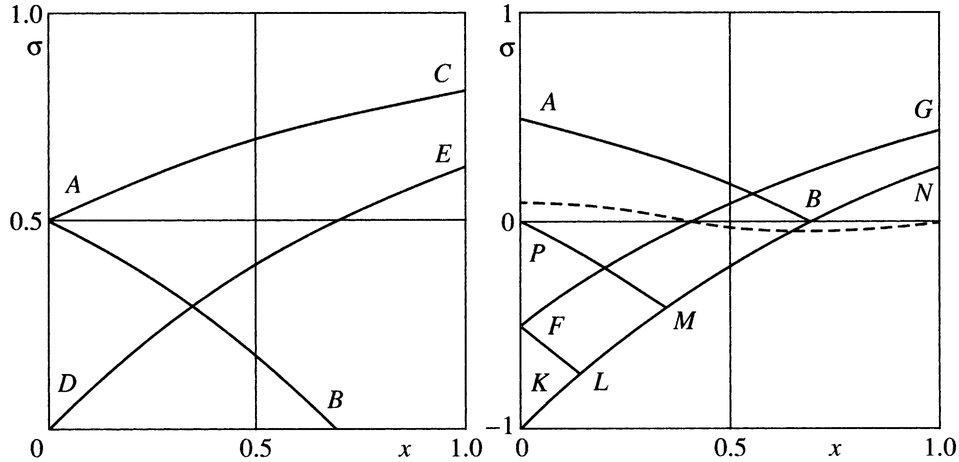

Fig. 3.

over the whole of the rod. The opening of the joint between the rod and the housing occurs over the whole length at the same time. Curve *AC* corresponds to $\sigma(0) = a_m/2$ but now during the unloading stage. After the temporary loss of contact between the rod and the housing, an unloading front is not formed. This effect has been discussed in the preceding section. The line *DE* describes the residual stress distribution after the loading – unloading half cycle, which corresponds to stretching.

The stress distribution during the transition to compression is shown on the right-hand side of Fig. 3 by the curves *FG* and *KN* for $\sigma(0) = -a_m/2$ and $\sigma(0) = -a_m$ respectively. On reducing the compression, a secondary unloading front is formed, the position of which during unloading to $\sigma(0) = -a_m/2$ and $\sigma(0) = 0$ is determined by the x coordinates of the points *L* and *M* respectively. The magnitude of the residual stresses after the first complete cycle of change in the loading is represented by the curve *PMN*. After the removal of the compressive load, both compressed and stretched segments remain in the rod. The residual displacements are shown by the dashed curve. On repeated stretching up to $\sigma(0) = a_m/2$, the stresses are again described by the curve *AB* which is duplicated on the right-hand side and the left-hand side of Fig. 3. The deformation process then acquires a periodic form.

The results of the calculation of the absorption coefficient in the case when $\theta = 0$ and of an initial compression $f = sf_0$, where $f_0 = -0.003$ are shown in Fig. 4. The curve $s = 0$ corresponds to a rod placed in a housing without precompression. In this case, the relation $\psi(\lambda)$ has the form characteristic of external friction: as λ increases, the dissipation of energy initially increases and, when there is a large amount of friction, it begins to decrease due to the rapid reduction in the displacements.

An analysis of the shape of the hysteresis loops helps to explain the form of the relation $\psi(\lambda)$ in the case of precompression. Such loops are shown in Fig. 5 for two levels of initial compression. The numbers with which they

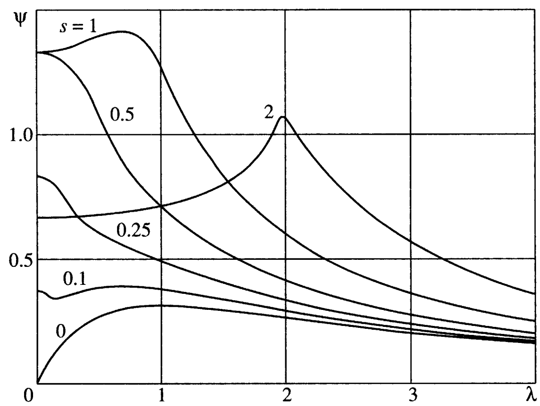

Fig. 4.

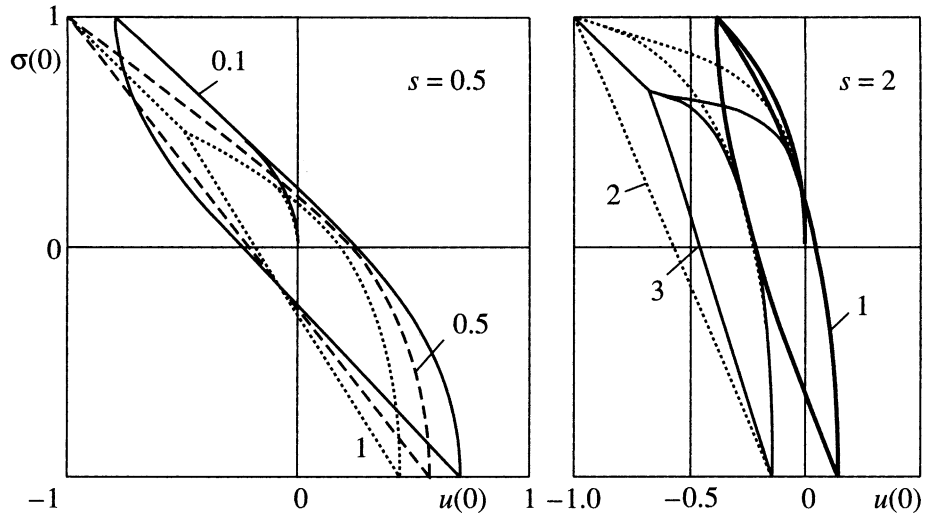

Fig. 5.

are labelled indicate the magnitude of the parameter λ. It essentially determines the sensitivity of the friction force to axial deformations of the rod. When it increases, that is, when there is an increase in this sensitivity, a more pronounced attenuation of the friction and, as a consequence, an increase in the displacement of the end face is observed for the same stretching forces. There is a loss of contact between the rod and the housing when λ increases further. It arises for the first time when $\lambda = s$ and one of the vertices of the loop appears at the point with coordinates $(-1, 1)$ in the $u(0) - \sigma(0)$ plane. Subsequently, as λ increases, an even greater part of the stretching phase of the rod takes place without contact with the housing. The area of the loop and, consequently, also the energy dissipated begin to decrease rapidly. When $s = 2$, such an evolution of the hysteresis loop served as the reason for the appearance of a cusped peak in the relation $\psi(\lambda)$. On the whole, initial compression affects the dissipation of energy more strongly than the value of λ, particularly for small values of this parameter.

The structural model of a rod enclosed in a housing enables one to simulate a wide range of problems associated with a friction interaction, and the results obtained can be used to solve them.

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